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Asymptotic Smoothing Effect for a Weakly Damped Nonlinear Schrodinger Equation in $\mathbb{T}^2$

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We prove that the global attractor for a weakly damped two-dimensional nonlinear Schrödinger equation in the usual energy space is in fact included and compact in a

smoothing effect; regularity of the attractor.

1. INTRODUCTION

We are interested in the long time behavior of solutions to the weakly damped nonlinear Schrödinger equations (NLS),

$$u_t + \alpha u + i\,\Delta u + ig(|u|^2)\,u = f. \tag{1.1}$$

Here the unknown u maps $\mathbb{T}_x^2 \times \mathbf{R}_t^+$ into $\mathbb{C}$, and we are given $\alpha > 0$, a damping parameter, and the external force f which does not depend on t. Throughout this article we will assume that f belongs to $L^2(\mathbb{T}^2)$, where $\mathbb{T}_x^2 = \mathbb{T}^2$ denotes the two-dimensional torus; in other words, we consider (1.1) on the unit square, with periodic boundary conditions.

We supplement (1.1) with the initial condition at $t = 0$,

$$u(0) = u_0 \in H^1(\mathbb{T}^2). \tag{1.2}$$

In this article we consider nonlinearities that are subcritical with respect to the H^1 norm. We are particularly interested in the focusing case, where the nonlinearity and the Laplacian produce competing effects (terms with opposite signs in the energy equation). However, we do not consider nonlinearities that allow solutions to blow up in H^1.

In fact, under suitable assumptions on g, it is well known that the solutions of (1.1)–(1.2) exist globally in time and that they are captured by an

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absorbing set in H^1. This is the starting point for proving the existence of a global attractor for the NLS (see [8, 13, 20]). On the other hand, it is well known that the NLS do not have a smoothing effect: any trajectory that starts from a point u_0 which belongs to the energy space H^1 remains in this space for all times, i.e., it does not enter in an $H^{1+\varepsilon}$ space for $\varepsilon > 0$. Despite this fact, we are able to prove that the damping provides to the NLS an asymptotic smoothing effect in the sense of [13, 14, 20]: actually, all solutions converge when time goes to the infinity to a compact subset of H^2. This means that the global attractor for the NLS is smooth, i.e., it only contains functions more regular than those in the energy space H^1.

Let us now make the assumptions on g more precise. We suppose that g is a smooth nonnegative function that maps $\mathbf{R}^+$ into $\mathbf{R}^+$, satisfies $g(0)=0$, and enjoys the growth condition

$$\text{there exists } p \text{ in } (1, 3) \text{ such that for } \xi \geqslant 1, \qquad g(\xi) = \xi^{(p-1)/2}. \tag{1.3}$$

This allows the usual nonlinearity $|u|^{p-1}u$, up to a regularization at $u=0$. One may also consider more general smooth functions g, such that $g(\xi)$ (respectively its derivatives) are bounded by $c\xi^{(p-1)/2}$ (respectively, bounded by the derivatives of $c\xi^{(p-1)/2}$) for large ξ.

Under these assumptions, we have the following existence result (see [1, 7] and the references therein for a proof).

Proposition 1.1. *The problem* (1.1)–(1.2) *has a unique solution*

$$u \in C([0, +\infty); H^1(\mathbb{T}^2)) \cap C^1([0, +\infty); H^{-1}(\mathbb{T}^2)),$$

and the mapping $S(t)\colon u_0 \mapsto u(t)$ *is continuous on* H^1.

Hence the semigroup $S(t)$ associated with the NLS and acting on H^1 is well defined. Moreover, the following statement, proved in [1], describes the dissipativity of $S(t)$.

Theorem 1.2. *The semigroup* $S(t)$ *has a compact global attractor* $\mathscr{A}$ *in* $H^1(\mathbb{T}^2)$.

We recall that a global attractor is a compact subset of the energy space that is invariant by the flow of the solutions and that attracts all the trajectories when time goes to infinity.

In this article, we aim to prove that this global attractor is in fact included and compact in a smaller energy space. Let us state our main result.

Theorem 1.3. *The global attractor* $\mathscr{A}$ *for the semigroup acting on* H^1, *defined by the NLS equation, is a compact subset of* H^2.

This result is sharp in the following sense: the global attractor, which contains all stationary solutions for (1.1), cannot be included in H^s for $s>2$ when we only assume that f belongs to L^2.

The issue of the regularity of the attractor is classical in the study of infinite-dimensional dissipative systems (we refer the reader to [20] for the general framework and for numerous applications). For the NLS, the regularity of the attractor was proved in [10] for the one-dimensional case, when x varies over a bounded interval of $\mathbf{R}$, with periodic boundary conditions. This result improved that of [8] where the existence of a global attractor for the weak topology in H^1 was proved (among other results) and that of [21] where the author established that this weak attractor is actually a strong attractor in the usual sense. In the one-dimensional case, it is also proved in [10] that if the external force is C^∞ then the attractor is also made of C^∞ functions. This result was recently improved in [17], where the authors prove that if the external force f belongs to some Gevrey space then $\mathscr{A}$ is also included in some Gevrey space.

We also mention that the issue of the regularity of the attractor for Korteweg–de Vries equations was addressed in [16] and for the dissipative Zakharov system in [12]. We refer the reader to [2] for the NLS in the one-dimensional case, when x varies over $\mathbf{R}$.

The first result for the two-dimensional NLS appears in [11], when the space variable x varies over $\mathbf{R}^2$. Multidimensional results are harder to prove; for instance, we have the technical difficulty inherited from the fact that H^1, the energy space, is not an algebra. In [11], we overcame these difficulties using the so-called Strichartz estimates that describe the dispersive nature of the NLS.

Unfortunately, these Strichartz estimates do not hold in the periodic case. Hence, we will use here instead the method introduced by Bourgain to handle the periodic Schrödinger equations (see [4, 9] and the references therein).

Let us point out that these ideas were also used, in the conservative case ($\alpha=0$ and $f=0$), to prove that global-in-time solutions for the nonlinear defocusing Schrödinger equations remain smooth, say, in H^s for some $s>1$, if the initial data belong to H^s (see [5] for instance). Our result is quite different, since we are in the focusing case and since we just assume that the initial data belong to H^1. Actually, we prove that the damping provides to the NLS a smoothing effect at $t=+\infty$.

This article is organized as follows: In Section 2 we introduce the Bourgain spaces that we need and we prove some nonlinear estimates in these spaces that will be used in the following; this section follows the framework in [9]. In Section 3 we recall some results concerning the existence of absorbing sets for the NLS, and we then complete these by proving that the solutions for the NLS remain, locally in time, bounded in some

Bourgain spaces. In Section 4 we introduce an auxiliary problem and then prove its well-posedness in H^2. Section 5 is devoted to establishing the asymptotic smoothing effect for the NLS; for that purpose, we begin with a long time comparison between the solutions to the NLS and to the auxiliary problem. We then prove our main result, which is the compactness of the global attractor in H^2.

We complete this introduction by setting some notation: we call H^1_x the usual Sobolev spaces $H^1(\mathbb{T}^2)$; L^p_x, $1 < p < +\infty$, stands for the space of measurable functions u such that $|u|^p$ is integrable over $\mathbb{T}^2$; L^∞_x is the usual space of essentially bounded functions. We will also use mixed space-time norms like $L^p_t H^s_x$, or norms introduced by Bourgain that we will define when needed.

For the sake of convenience, we will also write (1.1) in its abstract form, namely

$$u_t + Au = iF(u) + f, \tag{1.4}$$

where A stands for $A = i\Delta + \alpha$, and $F(u) = -g(|u|^2)u$.

2. PRELIMINARY RESULTS

2.1. *Bourgain Function Spaces*

In this section we describe function spaces that have been introduced by Bourgain for studying nonlinear dispersive evolution equations when the space variable x belongs to the n-dimensional torus $\mathbb{T}^n$. We follow here the lecture by Ginibre [9], and we refer the reader to the references therein.

Let $u(x, t)$ be a function that is periodic with respect to t and x. Its Fourier series reads

$$u(x, t) = \sum_{\xi \in Z^2} \sum_{\tau \in Z} \hat{u}(\xi, \tau)\, e^{2i\pi t\tau} e^{2i\pi x \cdot \xi}. \tag{2.1}$$

Let $H^{a,b}$ be the usual Sobolev space which contains the functions u such that

$$\|u\|^2_{H^{a,b}} = \sum_{\xi \in Z^2} \sum_{\tau \in Z} |\hat{u}(\xi, \tau)|^2 \,(1 + |\xi|^2)^a \,(1 + |\tau|^2)^b \tag{2.2}$$

is finite.

Let $U(t) = e^{-it\Delta}$ be the free Schrödinger group. We introduce $X^{a,b}$ as the space of functions u such that

$$\|u\|_{X^{a,b}} = \|U(-t)u\|_{H^{a,b}} < \infty. \tag{2.3}$$

We observe that if $b > \frac{1}{2}$, then $X^{a,b} \subset L_t^\infty H_x^a$; this holds true since $H_t^b \subset L_t^\infty$ and since $U(t)^{-1} = U(-t)$ is a unitary group in H_x^a. This fact can be used to construct a classical solution in H_x^1 (locally in time) to the NLS by performing a fixed point argument in $X^{1,b}$, $b > \frac{1}{2}$; (see [4, 9]).

2.2. *Nonlinear Estimates*

For later use we prove below some nonlinear estimates related to the action of F from $L_t^\infty H_x^1$ into Bourgain spaces. We give the complete proofs of these results for the convenience of the reader. The proofs follow intensively the guidelines in [9], wherein the author describes the method developed in [4]. But here we must overcome the difficulty that $F(u)$ has no polynomial structure like u^2 or $|u|^2 u = u^2 \bar{u}$. Let us mention that sharp results concerning the action of bilinear functionals on Bourgain spaces are available as well (see [15, 18]).

Before stating the first result, we introduce the following notation.

DEFINITION. Let a^+ (respectively, a^-) denote any fixed number b such that $b > a$ (respectively, $b < a$) and $|b - a|$ is close to 0.

PROPOSITION 2.1. *For any* $-3/8^- < -3/8$, *and for* ρ, $\sigma \geqslant 0$ *such that* $\rho + \sigma = 7/4^+$, *then*

$$\|F(u)\|_{X^{1,\,-3/8-}} \leqslant c\, \|u\|_{L_t^\infty H_x^\rho} \|u\|_{L_t^\infty H_x^\sigma} \|u\|_{L_t^\infty H_x^1}. \tag{2.4}$$

Proof of Proposition 2.1. We first observe that

$$F(u) = -h(u)\, u^2, \tag{2.5}$$

where

$$h(u) = \left(\int_0^1 g'(s\, |u|^2)\, ds \right) \bar{u}, \tag{2.6}$$

since $g(0) = 0$. Due to the growth hypotheses on g, h is a Lipschitz mapping from $\mathbb{R}^2$ into $\mathbb{R}^2$; therefore there exists c such that for any $\rho \in [0, 1]$,

$$\|h(u)\|_{H_x^\rho} \leqslant c\, \|u\|_{H_x^\rho}. \tag{2.7}$$

To prove (2.7) if $\rho = 0$ or 1 is easy. For the case $0 < \rho < 1$, we use the definition for H_x^ρ,

$$H_x^\rho = \left\{ u \in L_x^2;\ \iint_{\mathbb{T}^4} \frac{|u(x+r) - u(x)|^2}{|r|^{2(1+\rho)}}\, dx\, dr < \infty \right\}. \tag{2.8}$$

Hence (2.4) is a consequence of (2.7) and of Proposition 2.2 below.

PROPOSITION 2.2. *For any* $-3/8^- < -3/8$ *and for* ρ, $\sigma \geqslant 0$ *such that* $\rho + \sigma = 7/4^+$, *then*

$$\|u_1 u_2 u_3\|_{X^{1,\,-3/8^-}} \leqslant c \sum_{(i,\,j,\,k) = (1,\,2,\,3)} \|u_i\|_{L_t^\infty H_x^\rho} \|u_j\|_{L_t^\infty H_x^\sigma} \|u_k\|_{L_t^\infty H_x^1}. \tag{2.9}$$

Proof of Proposition 2.2

First step: Littlewood–Paley expansion of a function. For a function u in $H^1(\mathbb{T}^2)$ and for ξ in Z^2, let us set

$$\hat{u}(\xi) = \int_{\mathbb{T}^2} u(x)\, e^{-2i\pi x \cdot \xi}\, dx. \tag{2.10}$$

Throughout this article, we will refer to the support of $\hat{u}$ as the spectrum of u, denoted $\mathrm{Sp}(u)$. Here we consider only the Fourier series of u with respect to the space variable x. The Littlewood–Paley expansion of u reads

$$u = \sum_{l=0}^{\infty} u_l + u_{-1}, \tag{2.11}$$

where $u_{-1} = \hat{u}(0)$ and where

$$u_l(x) = \sum_{2^l \leqslant |\xi| < 2^{l+1}} \hat{u}(\xi)\, e^{2i\pi x \cdot \xi}. \tag{2.12}$$

In the following, we reserve the subscripts l, j, k for the terms involved in this Littlewood–Paley expansion. We denote by c a numerical constant that may vary from one line to another.

Second step: A paraproduct algorithm. Let u, v, w be in H_x^1. We then have

$$\begin{aligned} uvw = \sum_{j,\,k,\,l} u_j v_l w_k &= \sum_{j \leqslant l \leqslant k} u_j v_l w_k + \text{analogous terms} \\ &= u_{-1} v_{-1} w_{-1} + \sum_{j \leqslant l < k} u_j v_l w_k + \text{analogous terms}. \end{aligned} \tag{2.13}$$

It is easy to bound $u_{-1} v_{-1} w_{-1}$, therefore we omit the details.

Hence, we will focus on the boundedness of $S = \sum_{j \leqslant l < k} u_j v_l w_k$, the other terms being similar, up to a permutation of u, v, w.

Introducing

$$P_j u = \sum_{l < j} u_l, \tag{2.14}$$

we observe that

$$S=\sum_{l=-1}^{\infty}\sum_{k=l+1}^{\infty}(P_{l+1}u)\,v_l w_k=\sum_{l=-1}^{\infty}v_l(P_{l+1}u)(w-P_{l+1}w). \tag{2.15}$$

Using this kind of decomposition is classical for obtaining nonlinear functional estimates (see [3], for instance).

Following [9], we introduce a second decomposition for $w-P_{l+1}w$ as

$$w-P_{l+1}w=\sum_{\lambda}R_{\lambda,l}w, \tag{2.16}$$

where $\hat{R}_{\lambda,l}$ is supported in a ball $B(\lambda, 2^l)$ of center λ and radius 2^l. Actually, we consider a locally finite covering of $\mathbb{C}-B(0, 2^l)$ by balls of radius 2^l, such that there exists on m such that for each x in $\mathbb{C}-B(0, 2^l)$, x belongs to at most m balls $B(\lambda, 2^l)$. We choose a covering such that $B(\lambda, 2^l)\cap B(0, 2^{l-1})=\varnothing$ holds true.

At this stage, we proceed to the key argument.

Third step: A key lemma.

LEMMA 2.3. *Consider*

$$S=\sum_{l=-1}^{\infty}\sum_{\lambda}P_l Q_l R_{\lambda,l}, \tag{2.17}$$

where the spectra of $R_{\lambda,l}$, Q_l, P_l are included respectively in the ball $B(\lambda, 2^l)$, in the annulus $C_l=\{\xi;\, 2^l\leqslant|\xi|<2^{l+1}\}$, and in the ball $B_{l+1}=\{\xi;\, |\xi|<2^{l+1}\}$.

Set $P=\sum_{l=0}^{\infty}P_l$, $Q=\sum_{l=0}^{\infty}Q_l$, $R_l=\sum_{\lambda}R_{\lambda,l}$, $\|R\|_{H^1_x}=\sup_l\|R_l\|_{H^1_x}$; then for any $\rho, \sigma\geqslant 0$ such that $\rho+\sigma=7/4^+$,

$$\|S\|_{X^{1,-3/8-}}\leqslant c\,\|P\|_{L^\infty_t H^\rho_x}\,\|Q\|_{L^\infty_t H^\sigma_x}\,\|R\|_{L^\infty_t H^1_x}. \tag{2.18}$$

Remark. Observe that R_l has its spectrum outside $B(0, 2^{l-1})$. This fact will be used in the proof.

Proof of Lemma 2.3. The proof follows the guidelines in [9]. We first recall from this article the following statement.

PROPOSITION 2.4. *Let μ be a* fixed *positive number. Let f_j be such that its spectrum is included in a ball of* radius *$\mu 2^j$. Then there exists a $c=c_\mu$ that is independent of j, such that*

$$\|f_j\|_{L^4_{t,x}}\leqslant c2^{j/4}\,\|f_j\|_{X^{0,3/8+}} \tag{2.19}$$

$$\|f_j\|_{X^{0,-3/8-}}\leqslant c2^{j/4}\,\|f_j\|_{L^{4/3}_{t,x}}. \tag{2.20}$$

Proof of Proposition 2.4. For the convenience of the reader we sketch the main arguments of the proof, referring the reader to [9] or to [4] for more details. First of all, we observe that if the spectrum of u_j is included in B_j, then the following version of the Strichartz estimate holds true (see Proposition 3.114 in [4] or Proposition 4.2 in [9]):

$$\|U(t)\,u_j\|_{L^6_{x,t}} \leqslant c2^{j/3}\,\|u_j\|_{L^2_x}. \tag{2.21}$$

Here u_j is independent of t. Proceeding as in the Lemma 3.3 in [9], we thus obtain, for a function $f_j = f_j(x, t)$ whose spectrum is still included in B_j, that

$$\|f_j\|_{L^6_{x,t}} \leqslant c2^{j/3}\,\|f_j\|_{X^{0,(1/2)+}}. \tag{2.22}$$

Interpolating (2.22) with the identity $L^2_{x,t} = X^{0,0}$, we obtain

$$\|f_j\|_{L^4_{x,t}} \leqslant c2^{j/4}\,\|f_j\|_{X^{0,(3/8)+}}. \tag{2.23}$$

Hence (2.19) is proved as soon as we observe that (2.21) and thus (2.23) are invariant under translation of the spectrum of f_j (see Section 5 in [4] or the proof of Theorem 5.2 in [9]).

We complete the proof of Proposition 2.4 by observing that (2.20) is just the dual estimate for (2.19).

Remark. Observe that (2.19) and (2.20) depend only on the *diameter* of the spectrum of j and not on its location with respect to 0 (translation invariance).

We now proceed to the proof of (2.18). Due to the Minkowski inequality, we have

$$\|S\|_{X^{1,-3/8-}} \leqslant c \sum_l \left(\sum_j 4^j \left\| U^{-1} \sum_\lambda (P_l Q_l R_{\lambda,l})_j \right\|^2_{H_t^{-3/8} L^2_x} \right)^{1/2} \tag{2.24}$$

where $(U^{-1}S)_j = U^{-1}S_j$ denotes the part of $U^{-1}S$ whose spectrum is included in the annulus C_j.

Observe that there exists an absolute constant c (independent of l) such that if $|\lambda - \beta| \geqslant c$ then $P_l Q_l R_{\lambda,l}$ and $P_l Q_l R_{\beta,l}$ have disjoint spectra. On the other hand, the spectrum of $P_l Q_l R_{\lambda,l}$ is supported into a ball of radius $(|\lambda| + 5.2^l)$; then, since the mapping $u \to u_j$ is an orthogonal projector in L^2_x,

$$\|S\|_{X^{1,-3/8-}} \leqslant c \sum_{l} \left(\sum_{j,\lambda} 4^j \, \|U^{-1}(P_l Q_l R_{\lambda,l})_j\|^2_{H_t^{-3/8-} L_x^2} \right)^{1/2}$$

$$\leqslant c \sum_{l} \left(\sum_{\lambda} \left(\sum_{2^j < (|\lambda| + c2^l)} 4^j \right) \|U^{-1}(P_l Q_l R_{\lambda,l})\|^2_{H_t^{-3/8-} L_x^2} \right)^{1/2}$$

$$\leqslant c \sum_{l} \left(\sum_{\lambda} (|\lambda| + c2^l)^2 \, \|U^{-1}(P_l Q_l R_{\lambda,l})\|^2_{H_t^{-3/8-} L_x^2} \right)^{1/2}. \tag{2.25}$$

At this stage, we observe that $P_l Q_l R_{\lambda,l}$ has its spectrum included in a ball of width $c2^l$, independent of λ. Therefore, due to (2.20) and Hölder inequalities,

$$\|S\|_{X^{1,-3/8-}} \leqslant c \sum_{l} \left(2^{l/4} \left(\sum_{\lambda} (|\lambda| + c2^l)^2 \, \|P_l Q_l R_{\lambda,l}\|^2_{L^{4/3}_{x,t}} \| \right) \right)^{1/2}$$

$$\leqslant c \sum_{l} \left(2^{l/4} \, \|P_l Q_l\|_{L^4_{x,t}} \left(\sum_{\lambda} ((|\lambda| + c2^l)^2 \, \|R_{\lambda,l}\|^2_{L^2_{x,t}} \right) \right)^{1/2}. \tag{2.26}$$

Hence, since the covering is locally finite and since $\hat{R}_l$ is supported outside $B(0, 2^{l-1})$, we observe that

$$\sum_{\lambda} (|\lambda| + c2^l)^2 \, \|R_{\lambda,l}\|^2_{L^2_{x,t}} \leqslant c \sum_{\lambda} \|R_{\lambda,l}\|^2_{L^2_t H^1_x} \leqslant c \, \|R_l\|^2_{L^2_t H^1_x} \leqslant c \, \|R\|^2_{L^\infty_t H^1_x}. \tag{2.27}$$

For the last inequality, observe that $L^\infty_t \subset L^2_t$, since we are dealing with estimates local in time.

To complete the proof of (2.18), we will need the following classical inverse and enhanced Poincaré inequalities.

PROPOSITION 2.5. *There exists an absolute constant c such that if* $\mathrm{Sp}(y_l) \subset B_{l+1}$ *and* $\mathrm{Sp}(z_l) \cap B_l = \emptyset$, *then for* $\sigma \geqslant \rho$,

$$\|y_l\|_{L^\infty_x} \leqslant c(1+l)^{1/2} \, \|y_l\|_{H^1_x}, \tag{2.28}$$

$$\|y_l\|_{H^\sigma_x} \leqslant c2^{(\sigma-\rho)l} \, \|y_l\|_{H^\rho_x}, \tag{2.29}$$

$$\|z_l\|_{H^\rho_x} \leqslant c2^{-(\sigma-\rho)l} \, \|z_l\|_{H^\sigma_x}. \tag{2.30}$$

Proof of Proposition 2.5. We omit the proofs of (2.29) and (2.30) since they are easy (just expand y and z into their Fourier series). Inequality (2.28) can be established by using (2.29) and the so-called Brezis–Gallouët logarithmic inequality (see [6]). ∎

We now proceed to the majorization of the r.h.s. of (2.26); we have, for $\rho \leqslant 1$, applying (2.28)–(2.30),

$$\sum_l 2^{l/4} \|P_l Q_l\|_{L^4_{x,t}} \leqslant \sum_l 2^{l/4} \|P_l\|_{L^\infty_{x,t}} \|Q_l\|_{L^4_{x,t}}$$

$$\leqslant c \sum_l 2^{l/4}(1+l)^{1/2}\, 2^{(1-\rho)l} \|P_l\|_{L^\infty_t H^\rho_x} \|Q_l\|_{L^4_{x,t}}. \tag{2.31}$$

Observe that $\|P_l\|_{H^\rho_x} \leqslant \|P\|_{H^\rho_x}$ and that due to Sobolev embedding and (2.29),

$$\|Q_l\|_{L^4_x} \leqslant c\, \|Q_l\|_{H^{1/2}_x} \leqslant c2^{l/2}\, \|Q_l\|_{L^2_x}. \tag{2.32}$$

Therefore

$$\sum_l 2^{l/4} \|P_l Q_l\|_{L^4_{x,t}} \leqslant c\, \|P\|_{L^\infty_t H^\rho_x} \left(\sum_l (1+l)^{1/2}\, 2^{(7/4-\rho)l} \|Q_l\|_{L^4_t L^2_x} \right)$$

$$\leqslant c\, \|P\|_{L^\infty_t H^\rho_x} \left(\sum_l 2^{-o^+ l} \right)^{3/4} \left(\sum_l 2^{(7-4\rho)^+\, l} \|Q_l\|^4_{L^4_t L^2_x} \right)^{1/4}$$

$$\leqslant c\, \|P\|_{L^\infty_t H^\rho_x} \left(\int_{loc} \left(\sum_l 2^{(7/2-2\rho)^+\, l} \|Q_l\|^2_{L^2_x} \right)^2 dt \right)^{1/4}$$

$$\leqslant c\, \|P\|_{L^\infty_t H^\rho_x} \|Q\|_{L^4_t H^{(7/4-\rho)^+}_x}, \tag{2.33}$$

which implies (2.18) since we are dealing with estimates local in time. This concludes the proof of Lemma 2.3 and the proof of Proposition 2.2, since $\|R\|_{H^1_x} \leqslant \|w\|_{H^1_x}$. ∎

To complete this section, we state a result that will allow us to perform a fixed point argument in $X^{1,\, 1/2+\varepsilon}$ spaces.

Proposition 2.6. *There exists a c such that*

$$\|F(u) - F(v)\|_{X^{1,\,-3/8-}} \leqslant c(\|u\|^2_{L^\infty_t H^1_x} + \|v\|^2_{L^\infty_t H^1_x})\, \|u - v\|_{L^\infty_t H^1_x}. \tag{2.34}$$

Proof of Proposition 2.6. Observe that

$$F(v) - F(u) = (h(u) - h(v)) \cdot v^2 + h(u)(u - v)(u + v). \tag{2.35}$$

Then (2.34) follows from Proposition 2.2 and from the fact that h is a Lipschitz mapping on H^1_x; this last point is easy to check using the growth hypotheses on g. ∎

3. BOUNDED ABSORBING SETS

3.1. *Old Estimates*

We recall from [1] the following result, describing the dissipativity of the equation.

PROPOSITON 3.1. *There exists a bounded absorbing subset B_1 in $H^1(\mathbb{T}^2)$ satisfying that for any bounded subset B of $H^1(\mathbb{T}^2)$, there exists a $t_1 = t_1(B) > 0$ such that*

$$S(t)B \subset B_1 \qquad \textit{for} \quad t \geqslant t_1. \tag{3.1}$$

Moreover, there exists a K which depends on α and f such that for $t \geqslant t_1$,

$$\|u(t)\|_{H^1} + \|u_t(t)\|_{H^{-1}} \leqslant K. \tag{3.2}$$

Remark. Throughout this article, we will denote by C a constant which is independent of the data α, f, and we denote by K a constant depending on α, f. We allow C and K to vary from one line to another.

3.2. *New Estimates*

This section is devoted to proving that the trajectories which remain in the absorbing set B_1 are *locally in time* bounded in some Bourgain spaces.

In the conservative case, i.e., $\alpha = 0$ and $f = 0$, it is known that a solution for (1.1)–(1.2) in H^1_x can be obtained by performing a fixed point argument in $X^{1,1/2+}$ (see [4, 9]). We follow this method here.

Let $\frac{1}{2}^+$ be any fixed number, i.e., it satisfies $\frac{1}{2}^+ > \frac{1}{2}$ and $\frac{1}{2}^+ - \frac{1}{2}$ is small. Let $T > 0$ be a small number which depends on α, f, $\frac{1}{2}^+$ and that will be specified subsequently. We seek estimates that hold on time intervals of width T. We introduce a smooth cut-off function $\psi: \mathbb{R} \to [0, 1]$, whose support is included in $[\frac{-1}{4}, \frac{1}{4}]$, and which satisfies $\psi(t) = 1$ for $|t| \leqslant \frac{1}{8}$. For the sake of simplicity, we set $\psi_T(t) = \psi(\frac{t}{8T})$ and $\psi = \psi_1$.

A solution $u(t)$ for (1.1)–(1.2) is also a solution for $|t| \leqslant T$ to

$$U(-t)(u(t) - A^{-1}f) = \psi(t)\, e^{-\alpha t}(u_0 - A^{-1}f) \\ + i\psi_T(t) \int_0^t U(-s)\, e^{\alpha(s-t)} \psi(s)\, F(u(s))\, ds. \tag{3.3}$$

Let us state

PROPOSITION 3.2. *Assume $\frac{1}{2}^+ \in (\frac{1}{2}, \frac{1}{2} + \varepsilon]$, where $\varepsilon < 1/16$. There exist K, T which depend on α, f, ε such that the solution u for* (1.1), (1.2) *satisfies*

$$\|u - A^{-1}f\|_{X^{1,1/2+}_{\mathrm{loc}}} \leqslant K\, \|u_0 - A^{-1}f\|_{H^1_x}, \tag{3.4}$$

where the $X^{a,b}_{\mathrm{loc}}$ norm of a function v denotes the $X^{a,b}$ norm of the restriction of v to $[0, T]$.

Remark. One may wonder why we do not have an estimate for u in $X^{1,1/2+}_{\mathrm{loc}}$. In fact, since f is independent of t, the assertion

$$A^{-1}f \in X^{a,1/2+\rho}_{\mathrm{loc}} \Leftrightarrow f \in H^{a+2\rho-1}_x \tag{3.5}$$

holds true. Here it is not assumed that $A^{-1}f \in X^{1,1/2+\rho}_{\mathrm{loc}}$ for $\rho > 0$.

Proof of Proposition 3.2. For the sake of simplicity, we omit the subscript "loc" on $X^{a,b}_{\mathrm{loc}}$ spaces. We first observe that the affine space

$$\hat{X}^{1,1/2+} = \{v = A^{-1}f + u;\, u \in X^{1,1/2+}\}$$

is a complete metric space endowed with the distance $d(u, v) = \|u - v\|_{X^{1,1/2+}}$. Moreover $\hat{X}^{1,1/2+} \subset L^\infty_t H^1_x$, since for any t

$$\begin{aligned}\|v(t)\|_{H^1_x} &\leqslant \|v(t) - A^{-1}f\|_{H^1_x} + \|A^{-1}f\|_{H^1_x} \\ &\leqslant c(\|v - A^{-1}f\|_{X^{1,1/2+}_{\mathrm{loc}}} + \|f\|_{H^{-1}_x}).\end{aligned} \tag{3.6}$$

We now perform the fixed point argument: let u_0 be given in H^1_x and let $u(t)$ be in $\hat{X}^{1,1/2+}$. We define a mapping $\mathscr{T}$ as

$$U(-t)(\mathscr{T}(u(t)) - A^{-1}f) = \text{r.h.s of (3.3)}. \tag{3.7}$$

The first term on the r.h.s. of (3.3) is majorized by

$$\|\psi(t)\, e^{-\alpha t}(u_0 - A^{-1}f)\|_{H^{1/2+}_t H^1_x} \leqslant K(\alpha, \varepsilon)\, \|u_0 - A^{-1}f\|_{H^1_x}. \tag{3.8}$$

We chose R such that $\frac{R}{2} = $ r.h.s. of (3.8). We now handle the second term in the r.h.s of (3.3).

We first apply Lemma 3.2 in [9] which leads to

$$\left\|\psi_T(t)\, e^{-\alpha t} \int_0^t U(-s)\, \psi(s)\, e^{\alpha s} F(u(s))\, ds\right\|_{H^{1/2+}_t H^1_x} \leqslant cT^\varepsilon\, \|F(u)\|_{X^{1,-1/2+2\varepsilon}}. \tag{3.9}$$

Observe that since we are dealing with an estimate local-in-time, the $e^{\alpha s}$ term does not play a role and can be incorporated into the cut-off function ψ.

Assuming that ε is small enough to ensure $H^{-3/8-}_t \subset H^{-1/2+2\varepsilon}_t$, we then apply Proposition 2.1 to obtain

$$\|F(u)\|_{X^{1,-1/2+2\varepsilon}} \leqslant c\, \|u\|^3_{L^\infty_t H^1_x}. \tag{3.10}$$

We infer from (3.7) and (3.8) that if u belongs to the ball of center $A^{-1}f$ and radius R in $X^{1,(1/2)^+}$, then

$$\|\mathcal{T}(u(t)) - A^{-1}f\|_{X^{1,1/2+}} \leqslant R/2 + cT^{\varepsilon}(R + \|f\|_{H^{-1}})^3. \tag{3.11}$$

Therefore, if T is small enough, then $\mathcal{T}$ maps the ball of center $A^{-1}f$ and radius R into itself. Since proving that $\mathcal{T}$ is a contraction mapping is similar (we use Proposition 2.6 instead of Proposition 2.1) we omit the proof. Hence $\mathcal{T}$ has a unique fixed point u^* which satisfies

$$\|u^*(t) - A^{-1}f\|_{X^{1,1/2+}} \leqslant R = 2K(\alpha, \varepsilon)\, \|u_0 - A^{-1}f\|_{H^1_x}. \tag{3.12}$$

Thus since $u^*(t) = u(t)$ on $[0, T]$, the proof of Proposition 3.2 is complete. ∎

For later use, we infer from Propositions 3.1 and 3.2 the following result:

COROLLARY 3.3. *There exist T, K which depend only on α and f such that for any trajectory $u(t)$ that belongs to the absorbing ball for $t \geqslant t_1$, and for any time interval I included in $[t_1, \infty)$ and whose width is less than T, then*

$$\|u - A^{-1}f\|_{X^{1,1/2+}_{\mathrm{loc}(I)}} \leqslant K. \tag{3.13}$$

Proof of Corollary 3.3. In the formula above, the $X^{1,1/2^+}_{\mathrm{loc}(I)}$ norm of a function v denotes the $X^{1,1/2^+}$ norm of the restriction of v to I. Hence (3.13) follows from (3.2) and (3.4), which holds for $u_0 = u(t)$, for any $t \geqslant t_1$. ∎

4. THE AUXILIARY PROBLEM

4.1. *Definition*

Let $u(t)$ be a solution of (1.1)–(1.2) and let t_1 be its entrance time into the absorbing ball (see Proposition 3.1).

Let N be a positive number. We denote by

$$y = Pu = \sum_{\xi \in [-N, N]^2} \hat{u}(\xi)\, e^{2i\pi x \cdot \xi} \tag{4.1}$$

and

$$z = Qu = \sum_{\xi \notin [-N, N]^2} \hat{u}(\xi)\, e^{2i\pi x \cdot \xi} \tag{4.2}$$

the low-frequency part of u and the high-frequency part of u at level N, respectively.

We plan to approximate z at $t=+\infty$ by Z, that is the solution for

$$Z_t + AZ = iQF(y+Z) + Qf, \tag{4.3}$$

supplemented with initial condition

$$Z(t_1) = 0. \tag{4.4}$$

Remark. $u(t)$ being fixed, $y(t) = Pu(t)$ is datum for Eqs. (4.3)–(4.4).

4.2. *Local Existence Result*

A level N being fixed, it is easy to construct a local-in-time solution Z for (4.3), (4.4). For instance, we may perform a fixed point argument in $C([t_1, t_1+t]; H_x^2)$ for t small enough; actually, we just have to observe that F is a locally Lipschitz mapping on this space, that $Z(t_1)=0$ belongs to H_x^2, and that $y(t)$ remains bounded in H_x^2 due to (3.2) and to inverse inequality (2.29).

Hence there exists $T_{\max}$ such that (4.3)–(4.4) admit a unique solution Z defined on $[t_1, T_{\max})$ and such that either $T_{\max} = +\infty$ or

$$\|Z(t)\|_{H_x^2} \to \infty \qquad \text{when} \quad t \to T_{\max}, \quad t < T_{\max}. \tag{4.5}$$

The next subsection will prove that this solution is global in H_x^2.

4.3. *Global-in-Time Estimates*

PROPOSITION 4.1. *There exist K, N_0 which depend on the data α, f, and there exists a $\gamma \geqslant 1$ such that for any given $N \geqslant N_0$ the solution Z for* (4.3), (4.4) *satisfies, for any $t \geqslant t_1$,*

$$\|Z(t)\|_{H_x^1} + N^{-\gamma} \|Z(t)\|_{H_x^2} \leqslant K. \tag{4.6}$$

Remark. Actually, we could prove that $\gamma=1$ in (4.6). But because we do not need the optimal γ for our main theorem and because proving that $\gamma=1$ involves lengthly computations, we will just prove Proposition 4.1 as stated.

Proof of Proposition 4.1. For the sake of simplicity, we assume that $t_1=0$ throughout this proof.

First step: The local solution is global in H_x^1. We are going to prove that for $t < T_{\max}$,

$$\|Z(t)\|_{H_x^1} \leqslant K, \tag{4.7}$$

where K depends on α, f but is independent of N (it being understood that N is fixed large enough, $N \geqslant N_0$; we will make precise the definition of N_0 in the following).

Multiply (4.3) by $-\bar{Z}_t - \alpha \bar{Z}$ and integrate over $\mathbb{T}^2$ the imaginary part of the resulting equation. This leads to

$$\frac{1}{2}\frac{d}{dt} J(Z) + \alpha J(Z) = \alpha \,\mathrm{Im} \int_{\mathbb{T}^2} f\bar{Z} + \alpha \,\mathrm{Re} \int_{\mathbb{T}^2} g(|v|^2)\, v\bar{Z}$$

$$-\alpha \int_{\mathbb{T}^2} G(v) - \mathrm{Re} \int_{\mathbb{T}^2} g(|v|^2)\, v\bar{y}_t, \tag{4.8}$$

where $v = y + Z$, and where

$$J(Z) = \|Z\|^2_{\dot{H}^1_x} - \int_{\mathbb{T}^2} G(v) + 2 \,\mathrm{Im} \int_{\mathbb{T}^2} f\bar{Z}, \tag{4.9}$$

with $G(\xi) = \int_0^{|\xi|^2} g(s)\, ds$. Here $\|\cdot\|_{\dot{H}^1_x}$ stands for the L^2 norm of the gradient of Z. This norm and the usual H^1 norm define equivalent norms on QH^1_x, independent of N. We first observe that projecting (1.4) on PH^1_x leads to

$$y_t = -P(Au - f) + iPF(u). \tag{4.10}$$

We then integrate (4.8) on $[0, t]$, using

$$e^{\alpha t}\left(\frac{d}{dt} J + 2\alpha J\right) = \frac{d}{dt}(Je^{\alpha t}) + (Je^{\alpha t}),$$

and thus obtain

$$J(Z(t)) + \alpha \int_0^t e^{\alpha(s-t)} J(Z(s))\, ds$$

$$\leqslant J(0)\, e^{-\alpha t} + 2 \int_0^t e^{\alpha(s-t)} \left(\alpha \left|\mathrm{Im} \int_{\mathbb{T}^2} f\bar{Z}\right| + \alpha \left|\mathrm{Re} \int_{\mathbb{T}^2} g(|v|^2)\, v\bar{Z}\right|\right.$$

$$\left. + \alpha \left|\int_{\mathbb{T}^2} G(v)\right|\right) ds + {} + 2 \int_0^t |e^{\alpha(s-t)} \langle iPF(u), g(|v|^2) v\rangle|\, ds$$

$$+ 2 \left|\int_0^t e^{\alpha(s-t)} \langle P(Au - f), g(|v|^2) v\rangle\, ds\right|, \tag{4.11}$$

where $\langle\ ,\ \rangle$ denotes the L^2_x scalar product.

At this stage, we establish a lower bound for $J(Z)$. Due to the growth hypotheses on g, to the Sobolev embedding $H^s_x \subset L^{p+1}_x$, $1/(p+1) = (1-s)/2$, we have

$$\left| \int_{\mathbb{T}^2} G(v) \right| \leqslant c[\, \|y\|^{p+1}_{L^{p+1}_x} + \|Z\|^{p+1}_{L^{p+1}_x}]$$

$$\leqslant c[\, \|y\|^{p+1}_{H^s_x} + \|Z\|^{p+1}_{H^s_x}]. \tag{4.12}$$

We bound the y-term on the right-hand side of (4.12) using $H^1_x \subset H^s_x$ and the fact that u remains bounded in H^1_x (see (3.2)). We handle the Z-term by using the enhanced Poincaré inequality (2.30). Throughout this proof we may use these arguments in several places without notice. We thus obtain

$$J(Z) \geqslant \|Z\|^2_{\dot{H}^1_x} - K - \frac{c}{N^2} \|Z\|^{p+1}_{H^1_x} - 2\, \|f\|_{L^2_x} \|Z\|_{L^2_x}$$

$$\geqslant \frac{1}{2} \|Z\|^2_{H^1_x} - K - \frac{c}{N^2} \|Z\|^{p+1}_{H^1_x}. \tag{4.13}$$

Observe now that the same arguments allow us to bound the first three terms in the r.h.s of (4.8) as follows:

$$\left| \operatorname{Im} \int_{\mathbb{T}^2} f\bar{Z} \right| + \left| \operatorname{Re} \int_{\mathbb{T}^2} g(|v|^2)\, v\bar{Z} \right| + \left| \int_{\mathbb{T}^2} G(v) \right|$$

$$\leqslant \frac{1}{16} \|Z\|^2_{H^1_x} + K + K \frac{1}{N^2} \|Z\|^{p+1}_{H^1_x}. \tag{4.14}$$

We also have

$$\left| \operatorname{Re} \int_{\mathbb{T}^2} iPF(u)\, g(|v|^2)\, \bar{v}\, dx \right| \leqslant c\, \|F(u)\|_{L^{3/(3-p)}_x} \|F(v)\|_{L^{3/p}_x}$$

$$\leqslant K(1 + \|Z\|^p_{L^3_x})$$

$$\leqslant K\left(1 + \frac{1}{N^{2p/3}} \|Z\|^p_{H^1_x}\right); \tag{4.15}$$

here we have used also the following proposition (see [19] for a proof).

Proposition 4.2. *For $1 < p < +\infty$, there exists c_p which is independent of N such that*

$$\|Pu\|_{L^p_x} \leqslant c_p\, \|u\|_{L^p_x}. \tag{4.16}$$

We summarize (4.11)–(4.15) in

$$J(Z(t)) \leqslant J(0)\, e^{-\alpha t} + K \int_0^t e^{\alpha(s-t)} \left(\frac{1}{N^2} \|Z\|_{H_x^1}^{p+1} + \frac{1}{N^{2p/3}} \|Z\|_{H_x^1}^{p} + 1 \right) ds$$

$$+2 \left| \int_0^t e^{\alpha(s-t)} \langle P(Au-f),\, g(|v|^2)\, v \rangle \right|. \tag{4.17}$$

We introduce now a θ such that $t \leqslant \theta < T_{\max}$ and

$$M(\theta) = \|Z\|_{L^\infty([0,\,\theta];\, H_x^1)}. \tag{4.18}$$

We now aim to bound the last term on the r.h.s. of (4.17) using local-in-time estimates in Bourgain spaces. For this purpose we divide $[0, t]$ into $\bigcup_{k=1}^n I_k$ with $I_k = [k\tilde{T}, (k+1)\, \tilde{T}]$ and n chosen such that the width of I_k is less than T and larger than $T/2$, T being as in Corollary 3.3.

We then have

$$\left| \int_0^t e^{\alpha(s-t)} \langle P(Au-f), F(v) \rangle\, ds \right|$$

$$\leqslant \sum_{k=0}^n e^{\alpha(k\tilde{T}-t)} \left| \int_{I_k} e^{\alpha(s-k\tilde{T})} \langle P(Au-f), F(v) \rangle\, ds \right|$$

$$\leqslant c \sum_{k=0}^n e^{\alpha(k\tilde{T}-t)} \|Au-f\|_{X_{\mathrm{loc}(I_k)}^{-1,\,1/2-}} \|F(v)\|_{X_{\mathrm{loc}(I_k)}^{1,\,-1/2+}}. \tag{4.19}$$

Here we have used that P is bounded independent of N in Bourgain spaces and the fact that the $e^{\alpha(s-k\tilde{T})}$ does not play any role when we deal with local-in-time estimates.

On the one hand, due to (3.2) and to (3.4),

$$\|Au-f\|_{X_{\mathrm{loc}(I_k)}^{-1,\,1/2-}} \leqslant K. \tag{4.20}$$

On the other hand, applying Proposition 2.1, we have

$$\|F(v)\|_{X_{\mathrm{loc}(I_k)}^{1,\,-1/2+}} \leqslant c\, \|v\|_{L^\infty([0,\,\theta];\, H_x^1)} \|v\|_{L^\infty([0,\,\theta];\, H_x^{7/8+})}^2$$

$$\leqslant K(1+M(\theta)) \left(1 + \frac{M(\theta)^2}{N^{1/4-}} \right). \tag{4.21}$$

Using

$$\sum_{k=0}^n e^{\alpha(k\tilde{T}-t)} + \int_0^t e^{\alpha(s-t)}\, ds \leqslant K,$$

we infer from (4.13), (4.17)–(4.21), and the fact that $J(Z(0)) \leqslant 0$ that

$$\|Z(t)\|_{H_x^1}^2 \leqslant K_1 + K_2 \left(\frac{M(\theta)^{p+1}}{N^2} + \frac{M(\theta)^p}{N^{2p/3}} + (1+M(\theta)) \left(1 + \frac{M(\theta)^2}{N^{1/4-}}\right)\right). \tag{4.22}$$

Since (4.22) is valid for $t \leqslant \theta$, we may replace the l.h.s. of (4.22) by $M(\theta)^2$.

We now consider for $K_3 = K_1 + K_2$,

$$\Phi(M) = -M^2 + K_2 M + K_3 + K_2 \left(\frac{M^{p+1}}{N^2} + \frac{M^p}{N^{2p/3}} + \frac{M^3 + M^2}{N^{1/4-}} \right). \tag{4.23}$$

Then we have

$$\Phi(\sqrt{2K_3}) \leqslant -K_3 + K_2 \sqrt{2K_3} + K_4 \left(\frac{1}{N^2} + \frac{1}{N^{2p/3}} + \frac{1}{N^{1/4-}} \right). \tag{4.24}$$

Assuming without loss of generality that $K_3 \geqslant 2K_2(\sqrt{2K_3})$, i.e., K_3 that is large enough, we may chose $N_0 = N_0(K_3, K_4) = N_0(\alpha, f)$ such that if $N \geqslant N_0$ then $\Phi(\sqrt{2K_3}) < 0$. Since $\Phi(M(0)) = \Phi(0) = K_3 > 0$, since $\Phi(+\infty) = +\infty$, we must have

$$M(\theta) \leqslant \sqrt{2K_3} \tag{4.25}$$

for all θ in $[0, T_{\max})$. This concludes the proof of the first step.

Second step: The solution is global in H_x^2. Let us set ∂ for either $\partial/\partial x_1$ or $\partial/\partial x_2$, and $w = \partial Z$. Differentiating (4.3), we observe that w is a solution for

$$w_t + Aw = iQF'(v) \cdot (w + \partial y) + Q\,\partial f, \tag{4.26}$$

supplemented with the initial condition $w(0) = 0$. We recall that v stands for $v = y + Z$, and that we have

$$F'(v) \cdot w = -g(|v|^2)\, w - 2g'(|v|^2)\, \mathrm{Re}(\bar{v} w)\, v. \tag{4.27}$$

Multiply now (4.26) by $-\bar{w}_t - \alpha \bar{w}$ and integrate the imaginary part of the resulting equation over $\mathbb{T}^2$. This leads to

$$\frac{1}{2}\frac{d}{dt}J(w)+\alpha J(w)$$

$$=\left\{-\mathrm{Re}\int_{\mathbb{T}^2}\bar{v}_t vg'(|v|^2)\,|w|^2-2\,\mathrm{Re}\int_{\mathbb{T}^2}g'(|v|^2)\,\mathrm{Re}(\bar{v}w)\,w\bar{v}_t\right.$$

$$\left.-2\,\mathrm{Re}\int_{\mathbb{T}^2}g''(|v|^2)\,\mathrm{Re}(\bar{v}w)^2\,v\bar{v}_t\right\}$$

$$\times\left\{-\mathrm{Re}\int_{\mathbb{T}^2}\frac{d}{dt}\left[g(|v|^2)\,\partial y\right]\cdot\bar{w}-2\,\mathrm{Re}\int_{\mathbb{T}^2}\frac{d}{dt}\left[g'(|v|^2)\,\mathrm{Re}(\bar{v}\,\partial y)v\right]\cdot\bar{w}\right\}$$

$$+\alpha\,\mathrm{Re}\int_{\mathbb{T}^2}F'(v)\cdot\partial y\bar{w}-\alpha\,\mathrm{Im}\int_{\mathbb{T}^2}f\,\partial\bar{w}, \tag{4.28}$$

where

$$J(w)=\|w\|^2_{\dot{H}^1}+\mathrm{Re}\int_{\mathbb{T}^2}F'(v)\cdot w\bar{w}+2\,\mathrm{Re}\int_{\mathbb{T}^2}F'(v)\cdot\partial y\bar{w}$$

$$-2\,\mathrm{Im}\int_{\mathbb{T}^2}f\,\partial\bar{w}. \tag{4.29}$$

To begin with, we prove the coercivity of $J(w)$ in QH^1.

LEMMA 4.3. *There exist K, N_0, depending on the data α, f, such that for fixed $N\geqslant N_0$, $\forall t\geqslant 0$, $\forall w\in QH^1$,*

$$J(w)\geqslant\tfrac{1}{2}\,\|w\|^2_{H^1}-K. \tag{4.30}$$

Proof of Lemma 4.3. We recall that we have assumed $t_1=0$. Due to the growth hypotheses on g we have

$$\left|\mathrm{Re}\int_{\mathbb{T}^2}F'(v)\cdot w\bar{w}\right|\leqslant c\,\|v\|^{p-1}_{L^{2p-2}_x}\,\|w\|^2_{L^4_x}$$

$$\leqslant c\,\|v\|^{p-1}_{L^{2p-2}_x}\,\|w\|_{L^2_x}\,\|w\|_{H^1_x}, \tag{4.31}$$

due to $H^{1/2}_x\subset L^4_x$ and to interpolation inequality. We observe that $v=y+Z$ remains in a bounded set of $H^1_x\subset L^{2p-2}_x$, independent of N; this is due to (3.2) for $y=Pu$ and (4.7) for Z. Then, by applying Proposition 2.5, we obtain that the r.h.s. of (4.31) can be majorized by $(K/N)\,\|w\|^2_{H^1_x}$.

We now bound the last two terms in the r.h.s. of (4.29). We first have, as in the proof of (4.31),

$$\begin{aligned}\left|\operatorname{Re}\int F'(v)\cdot\partial y\,\bar{w}\right| &\leqslant K\,\|\partial y\|_{L_x^4}\,\|w\|_{L_x^4}\\ &\leqslant K\,\|\partial y\|_{H_x^{1/2}}\,\|w\|_{H_x^{1/2}}\\ &\leqslant K\,\|\partial y\|_{L_x^2}\,\|w\|_{H_x^1},\end{aligned}\tag{4.32}$$

due to $H_x^{1/2}\subset L_x^4$, the inverse inequality for ∂y, and the enhanced Poincaré inequality for w (see Proposition 2.5.).

We easily infer from (4.32) and the fact that ∂y remains bounded in L_x^2 that the last two terms on the r.h.s. of (4.29) can be bounded by $K+\frac{1}{24}\,\|w\|_{H_x^1}^2$.

To complete the proof of Lemma 4.3 is then straightforward. ∎

We then proceed to the majorization of the r.h.s. of (4.28); for this purpose we will use in several places (and without notice) inverse inequalities to bound the y-terms and the enhanced Poincaré inequalities for the Z-terms. We will also use the fact that $v=y+Z$ remains bounded in H_x^1 for $t\geqslant t_1=0$ (see (3.2) and (4.7)). We begin with

$$\begin{aligned}&\left|\operatorname{Im}\int f\,\partial\bar{w}\right|+\left|\operatorname{Re}\int F'(v)\cdot\partial y\,\bar{w}\right|\\ &\qquad\leqslant(\|f\|_{L_x^2}+\|\partial y\|_{L_x^2}\,\|F'(v)\|_{L_x^4})\,\|w\|_{H_x^1}\leqslant K\,\|w\|_{H_x^1}.\end{aligned}\tag{4.33}$$

We now bound the terms involved in the second bracket on the r.h.s. of (4.28); we just indicate how to handle the first one, since the majorization of the second one is similar.

We first have

$$\begin{aligned}\left|\operatorname{Re}\int\partial\bar{y}_t\,g(|v|^2)\,w\right| &\leqslant\|g(|v|^2)\|_{L_x^4}\,\|\partial y_t\|_{L_x^2}\,\|w\|_{L_x^4}\\ &\leqslant K(N^2\,\|y_t\|_{H_x^{-1}})\,N^{-1/2}\,\|w\|_{H_x^1}\\ &\leqslant KN^{3/2}\,\|w\|_{H_x^1},\end{aligned}\tag{4.34}$$

since $y_t=Pu_t$ remains bounded in H_x^{-1} (see (3.2)).

We postpone the majorization of

$$\operatorname{Re}\int\bar{v}_t v g'(|v|^2)\operatorname{Re}(\bar{w}\,\partial y)$$

and integrate now (4.28) on $[0, t]$; using (4.33), (4.34) we thus obtain

$$J(w(t)) \leqslant J(0)\, e^{-\alpha t} + \left| \int_0^t e^{\alpha(s-t)} \langle v_t,\, g'(|v|^2) v\, |w|^2 \rangle\, ds \right|$$
$$+ \left| \int_0^t e^{\alpha(s-t)} \langle v_t,\, g'(|v|^2)\, v\, \mathrm{Re}(\bar{w}\, \partial y) \rangle\, ds \right|$$
$$+ \text{analogous terms} + KN^{3/2} \int_0^t e^{\alpha(s-t)}\, \|w(s)\|_{H_x^1}\, ds. \tag{4.35}$$

We now proceed as in the first step, observing that

$$v_t = y_t + Z_t = (-P(Au-f) - Q(Av-f)) + (iPF(u) + iQF(v))$$
$$= (\text{terms locally bounded in Bourgain spaces}) + (\text{polynomial terms}). \tag{4.36}$$

The polynomial terms are bounded as follows:

$$|\langle iPF(u) + iQF(v),\, g'(|v|^2)\, v\, |w|^2 \rangle|$$
$$\leqslant c(\|F(u)\|_{L_x^4} + \|F(v)\|_{L_x^4})\, \|g'(|v|^2) v\|_{L_x^4}\, \|w\|^2_{L_x^4}$$
$$\leqslant K\, \|w\|_{H_x^1}\, \|w\|_{L_x^2} \leqslant \frac{K}{N}\, \|w\|^2_{H_x^1}. \tag{4.37}$$

For the other terms we proceed as in (4.19)–(4.20). Set $[0, t] = \bigcup_{k=1}^n I_k$ where the width of I_k is less than T and larger than $T/2$, T being as in Corollary 3.3. Then

$$\left| \int_0^t e^{\alpha(s-t)} \langle P(Au-f) + Q(Av-f),\, g'(|v|^2)\, v\, |w|^2 \rangle \right|$$
$$\leqslant K \sup_k ((\|Au-f\|_{X^{-1,\,1/2-}_{\mathrm{loc}(I_k)}} + \|Av-f\|_{X^{-1,\,1/2-}_{\mathrm{loc}(I_k)}})$$
$$\cdot\, \|g'(|v|^2)\, v\, |w|^2\|_{X^{1,\,-1/2+}_{\mathrm{loc}(I_k)}}). \tag{4.38}$$

We first observe

$$\|Au-f\|_{X^{-1,\,1/2-}_{\mathrm{loc}(I_k)}} + \|Av-f\|_{X^{-1,\,1/2-}_{\mathrm{loc}(I_k)}} \leqslant K; \tag{4.39}$$

this is valid, due to Corollary 3.3 and

Lemma 4.4. *There exist K, T which depend only on α, f such that if Z solves* (4.3), (4.4) *then for any time interval $I \subset [t_1, +\infty)$ whose width is less than T,*

$$\|Av-f\|_{X^{-1,\,1/2-}_{\mathrm{loc}(I)}} \leqslant K. \tag{4.40}$$

Proof of Lemma 4.4. Just copy the proof of Corollary 3.3, since Z remains bounded in H_x^1 as well (see (4.7)). ∎

Then, applying Proposition 2.1 and observing that $v \to g'(|v|^2)v$ is a bounded map on H_x^1, we have

$$\|g'(|v|^2)\, v\, |w|^2\|_{X^{1,-1/2+}_{\mathrm{loc}(I_k)}} \leqslant c\, \|g'(|v|^2)v\|_{L_t^\infty H_x^1} \|w\|_{L_t^\infty H_x^1} \|w\|_{L_t^\infty H_x^{3/4+}}$$

$$\leqslant \frac{K}{N^{1/4-}} \|w\|^2_{L_t^\infty H_x^1}. \tag{4.41}$$

The same ideas lead to

$$\left|\int_0^t e^{\alpha(s-t)} \langle v_t,\, g'(|v|^2)\, v\, \mathrm{Re}(\bar{w}\, \partial y)\rangle\, ds\right| \leqslant K\, \|w\|_{L_t^\infty H_x^1} \|\partial y\|_{L_t^\infty H_x^1}$$

$$\leqslant KN\, \|w\|_{L_t^\infty H_x^1}. \tag{4.42}$$

We now infer from (4.30), (4.35), and (4.37)–(4.42) that for $t < T_{\max}$

$$\|w(t)\|^2_{H_x^1} \leqslant K_1 + \frac{K_2}{N^{1/4-}} \|w\|^2_{L_t^\infty H_x^1} + K_3 N^{3/2}\, \|w\|_{L_t^\infty H_x^1}$$

$$\leqslant K_4 N^3 + \frac{K_2}{N^{1/4-}} \|w\|^2_{L_t^\infty} + \frac{1}{2} \|w\|^2_{L_t^\infty H_x^1}. \tag{4.43}$$

Taking the supremum on $t < T_{\max}$ in the l.h.s. of (4.43) and assuming that N_0 is such that $4K_2 \leqslant (N_0)^{1/4}$ complete the proof of the proposition. ∎

5. THE MAIN RESULT

5.1. *Large Time Comparison Between the Solutions of the Two Problems*

Let $u(t)$ be any solution of (1.1)–(1.2), and let t_1 be its entrance time into the absorbing set B_1. For a given $N \geqslant N_0$, N_0 being as in Proposition 4.1, we introduce a $Z(t)$ that is the solution of (4.3)–(4.4). We will compare $u(t)$ to $v(t) = y(t) + Z(t)$, where $y(t) = Pu(t)$, for large t's. Let us now state a result.

PROPOSITION 5.1. *There exist K, N_0 depending on the data α, f, such that for any given $N \geqslant N_0$, for $t \geqslant t_1$,*

$$\|z(t) - Z(t)\|_{H^1} \leqslant K \cdot e^{-\alpha(t-t_1)}. \tag{5.1}$$

Proof of Proposition 5.1. Let us set $\chi = u - v = z - Z$. Due to (1.1), (1.2), (4.3), and (4.4), χ is a solution for

$$\begin{aligned} &\chi_t + A\chi = iQF(y+z) - iQF(y+Z), \\ &\chi(t_1) = z(t_1). \end{aligned} \tag{5.2}$$

On the other hand, we have the a.e. in x equation

$$F(y+z) - F(y+Z) = \int_0^1 F'(v + \tau\chi) \cdot \chi \, d\tau. \tag{5.3}$$

In order to simplify the notation, we set

$$\{m\} = \int_0^1 m(\tau) \, d\tau \tag{5.4}$$

and $\theta = v + \tau\chi$. We now multiply (5.2) by $-\bar{\chi}_t - \alpha\bar{\chi}$, and then integrate the imaginary part of the resulting equation over $\mathbb{T}^2$ to obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt} J(\chi) + \alpha J(\chi) = \Bigg\{ &-\frac{1}{2}\int_{\mathbb{T}^2} \frac{\partial}{\partial t}(g(|\theta|^2)) \, |\chi|^2 - 2\int_{\mathbb{T}^2} g'(|\theta|^2) \operatorname{Re}(\bar{\theta}_t\chi) \operatorname{Re}(\bar{\theta}\chi) \\ &- \int_{\mathbb{T}^2} \frac{\partial}{\partial t}(g'(|\theta|^2)) \operatorname{Re}(\bar{\theta}\chi)^2 \Bigg\}, \end{aligned} \tag{5.5}$$

where

$$J(\chi) = \|\chi\|^2_{\dot{H}^1} + \left\{ \operatorname{Re} \int_{\mathbb{T}^2} F'(\theta) \, \chi\bar{\chi} \right\}. \tag{5.6}$$

Let us recall that the $\dot{H}^1$ norm of χ stands for the L^2 norm of the gradient of χ. To begin with, we prove the coercivity of $J(\chi)$ in QH^1_x.

LEMMA 5.2. *There exist K, N_0 depending on the date α, f, such that for fixed $N \geqslant N_0$, $\forall t \geqslant t_1$, $\forall \chi \in QH^1_x$,*

$$J(\chi) \geqslant \tfrac{1}{2} \|\chi\|^2_{H^1_x}. \tag{5.7}$$

Proof of Lemma 5.2. Due to the growth hypotheses on g, we have

$$\left| \left\{ \operatorname{Re} \int_{\mathbb{T}^2} F'(\theta) \cdot \chi\bar{\chi} \right\} \right| \leqslant c\Big(\sup_{\tau \in [0,1]} \|\theta\|^{p-1}_{L^{2p-2}_x}\Big) \, \|\chi\|_{L^2_x} \|\chi\|_{H^1_x}. \tag{5.8}$$

On the one hand, since $\theta = \tau u + (1-\tau) v = y + \tau z + (1-\tau) Z$ and since y, z, Z remain for $t \geqslant t_1$ in a bounded set of $H^1_x \subset L^{2p-2}_x$ (this set being independent of N; see (3.2) and (4.7)), we obtain that

$$\sup_{\tau \in [0,1]} \|\theta\|^{p-1}_{L^{2p-2}_x} \leqslant K. \tag{5.9}$$

On the other hand, we use the enhanced Poincaré inequality (2.30) to infer from (5.8) and (5.9) that

$$\left| \left\{ \operatorname{Re} \int_{\mathbb{T}^2} F'(\theta) \cdot \chi \bar{\chi} \right\} \right| \leqslant \frac{K}{N} \|\chi\|^2_{H^1_x}. \tag{5.10}$$

Hence (5.7) holds true as soon as N_0 is chosen large enough. ∎

We now integrate (5.5) on $[0, t]$, proceeding as in (4.11), and thus obtain

$$\begin{aligned} J(\chi(t)) &+ \alpha \int_{t_1}^{t} e^{\alpha(s-t)} J(\chi(s))\, ds \\ &\leqslant J(\chi(t_1))\, e^{\alpha(t_1 - t)} + \left\{ e^{\alpha(t_1-t)} \left| \int_{t_1}^{t} e^{\alpha(s-t_1)} \langle \theta_t, g'(|\theta|^2) \operatorname{Re}(\bar{\theta}\chi)\chi\rangle\, ds \right| \right\} \\ &\quad + \text{analogous terms}; \end{aligned} \tag{5.11}$$

here we just indicate how to bound the second term on the r.h.s. of (5.5). We omit the majorization of the other ones that is similar.

For τ in $[0, 1]$, we introduce the splitting

$$\begin{aligned} \theta_t &= \tau u_t + (1-\tau)\, v_t = y_t + \tau z_t + (1-\tau)\, Z_t \\ &= -(P(Au - f) + \tau Q(Au - f) + (1-\tau)\, Q(Av - f)) \\ &\quad + i(PF(u) + \tau QF(u) + (1-\tau)\, QF(v)). \end{aligned} \tag{5.12}$$

On the one hand, proceeding as in (4.37), we obtain that

$$\begin{aligned} &\sup_\tau |\langle i(PF(u) + \tau QF(u) + (1-\tau)\, QF(v)), g'(|\theta|^2) \operatorname{Re}(\bar{\theta}\chi)\chi\rangle| \\ &\quad \leqslant \frac{K}{N} \|\chi\|^2_{H^1_x}. \end{aligned} \tag{5.13}$$

Therefore this term can be bounded by $(\alpha/64)\, J(\chi)$ if N is large enough as assumed.

On the other hand, proceeding as in (4.38)–(4.39), we introduce a splitting $[t_1, t] = \bigcup_{k=1}^n I_k$ such that the width of I_k is less than T and larger than $T/2$ (T being as in Corollary 3.3 and Lemma 4.4), and thus obtain

$$\sup_\tau \left| \int_{t_1}^{t} e^{\alpha(s-t_1)} (\langle (P(Au-f) + \tau Q(Au-f) \right.$$

$$\left. + (1-\tau)\, Q(Av-f)),\ g'(|\theta|^2)\, \mathrm{Re}(\bar{\theta}\chi)\,\chi\rangle\, ds \right|$$

$$\leqslant K \sup_k (\|Au-f\|_{X^{-1,\,1/2+}_{\mathrm{loc},\,I_k}} + \|Av-f\|_{X^{-1,\,1/2+}_{\mathrm{loc},\,I_k}})\, \|g'(|\theta|^2)\, \mathrm{Re}(\bar{\theta}\chi)\chi\|_{X^{1,\,-1/2+}_{\mathrm{loc},\,I_k}}$$

$$\leqslant \frac{K}{N^{1/4-}} \|\chi\|^2_{L^\infty([t_1,\,t];\,H^1_x)}. \tag{5.14}$$

We now infer from (5.11)–(5.14) that for $t \in [t_1, t_1 + T]$,

$$J(\chi(t))\, e^{\alpha t} \leqslant J(\chi(t_1))\, e^{\alpha t_1} + \frac{K}{N^{1/4-}} \|\chi\|^2_{L^\infty([t_1,\,t_1+T];\,H^1_x)}\, e^{\alpha t_1}. \tag{5.15}$$

We easily deduce from (5.15) and (5.7) that (5.1) holds true.
This completes the proof of Proposition 5.1. ∎

5.2. *Proof of the Main Theorem.*

We now proceed to the proof of Theorem 1.3. To begin with, we prove that $\mathscr{A}$ is a bounded subset of H^2_x. Let a be in $\mathscr{A}$ and let N_0 be as in Proposition 4.1 and Proposition 5.1. Let N be a given integer $\geqslant N_0$, say, $N = N_0 + 1$ for instance. We first observe that, for $Q = Q_N$,

$$a \in H^2_x \Leftrightarrow Qa \in H^2_x. \tag{5.16}$$

Let m be a positive integer and let Z^m be the solution for

$$Z^m_t + AZ^m = iQF(y + Z^m) + Qf, \tag{5.17}$$

where $y(t) = Pu(t) = PS(t)a$, supplemented with initial condition at $t_1 = -m$,

$$Z^m(-m) = 0. \tag{5.18}$$

Since $u(t) = S(t)a \in \mathscr{A}$, $\forall t \in \mathbb{R}$, then Propositions 4.1 and 5.1 apply and $Z^m(0)$ satisfies

$$\|Z^m(0)\|_{H^2_x} \leqslant K \cdot N^\gamma \tag{5.19}$$

and

$$\|Z^m(0) - Qa\|_{H^1_x} \leqslant K \cdot e^{-\alpha m}. \tag{5.20}$$

Actually, (5.19) is valid since (4.6) holds true for $Z^m(t)$ for any $t \geqslant -m$, and (5.20) is just (5.1) with $t = 0$ and $t_1 = -m$. We can extract from $Z^m(0)$

a subsequence $Z^{m'}(0)$ that is weakly convergent in QH_x^2. Due to (5.20), the limit of this subsequence is Qa. Therefore

$$\|Qa\|_{H_x^2} \leqslant K \cdot N^\gamma, \tag{5.21}$$

and $\mathscr{A}$ is included and bounded in H_x^2, since Pa also satisfies (5.21), due to the inverse inequality (2.29).

The remainder of the proof is devoted to establishing the compactness of $\mathscr{A}$ into H_x^2. Since the proof is classical, using an argument due to J. Ball, we omit it and refer the reader to [11], where the same argument was used. ∎

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